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Compact Operators in the Commutant of a Contraction*

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1. All Hilbert spaces considered in this paper are assumed to be complex and separable. All operators are assumed to be bounded and linear. A contraction is a (bounded linear) operator of norm less than or equal to one.

Let T be a contraction on a Hilbert space and let U be the minimal strong unitary dilation of T. In case U is the bilateral shift of multiplicity one, Sarason [16, Theorem 1] characterized the commutant of T in terms of the commutant of U. In the context of this theorem, he also gave a necessary and sufficient condition that an operator in the commutant of T be compact [16, Theorem 2]. Subsequently, Sz.-Nagy and Foiaş characterized the commutant of an arbitrary contraction in terms of its minimal strong unitary dilation [20] (see also Refs. [19] and [4]). The prupose of this paper is to present a generalization of Sarason's Theorem 2 in the context of the Sz.-Nagy–Foiaş theorem. In a sense that will be made clear later, our result is formally the same as Sarason's and our proof follows his quite explicitly. However, our proof is technically rather complicated and much of the paper is devoted to the presentation of the technical machinery which we need.

In Section 2 we establish the setting of our theorem (Theorem III below) characterizing compact operators in the commutant of a contraction and state it. As we will see, our theorem applies to a slightly restricted class of contractions. In Section 3 we will establish the generality of our theorem by showing that "in the most interesting cases" when a contraction T commutes with a compact operator,

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T belongs to this restricted class. Sections 4 and 5 contain the bulk of the technical material which we will need to prove Theorem III. In Section 6 we present the proof. Section 7 is devoted to two immediate consequences of our main theorem. The first characterizes compact operators in terms of the Sz.-Nagy–Foiaş structure theory for operators while the second establishes a necessary and sufficient condition that a one-parameter semigroup of contractions be eventually compact. In Section 8 we discuss the relationship between our main theorem and a class of operators which we call generalized Hankel operators. Some of the results presented in this section were discovered independently by Page [14].

2. In this section we describe a functional representation for a certain class of contractions and state our main theorem. Our objective is primarily to establish notation and to state some of the basic results which we will use later. For the most part the material which we are discussing here may be found in either Ref. [8] or Ref. [18].

If $\mathscr{H}$ is a Hilbert space, then $\mathscr{L}(\mathscr{H})$ will denote the algebra of all (bounded linear) operators on $\mathscr{H}$. If T is in $\mathscr{L}(\mathscr{H})$ and $\mathscr{M}$ is a subspace invariant under T, then the operator in $\mathscr{L}(\mathscr{M})$ obtained by restricting T to $\mathscr{M}$ will be denoted by $T \mid \mathscr{M}$. The unit circle in the complex plane will be denoted by $\mathbf{T}$, Δ will denote the open unit disk, and m will denote normalized Lebesgue measure on $\mathbf{T}$.

Let $\mathscr{E}$ be a Hilbert space; we do not exclude the possibility that the dimension of $\mathscr{E}$ is finite. The space $\mathbf{L}^2_{\mathscr{E}}$ is defined to be the set of all (equivalence classes of) measurable norm-square integrable, $\mathscr{E}$-valued functions defined on $\mathbf{T}$. When endowed with the inner product defined by the equation

$$(f(\cdot), g(\cdot))_{\mathbf{L}^2_{\mathscr{E}}} = \int_{\mathbf{T}} (f(z), g(z))_{\mathscr{E}}\, dm, \quad f(\cdot), g(\cdot) \in \mathbf{L}^2_{\mathscr{E}},$$

$\mathbf{L}^2_{\mathscr{E}}$ becomes a (separable) Hilbert space. (For the most part, unless some confusion may result, we will omit subscripts on norms, inner products, and the like.) The subspace of $\mathbf{L}^2_{\mathscr{E}}$, consisting of those functions whose Fourier coefficients of negative index vanish, will be denoted by $\mathbf{H}^2_{\mathscr{E}}$. Each function in $\mathbf{H}^2_{\mathscr{E}}$ admits a natural analytic continuation into Δ.

A function $A(\cdot)$ from $\mathbf{T}$ into $\mathscr{L}(\mathscr{E})$ is called weakly measurable in case the complex-valued function $z \to (A(z)x, y)$ is measurable for every x and y in $\mathscr{E}$. If $A(\cdot)$ is weakly measurable then the real-valued function $z \to \| A(z)\|$ is measurable and the space of all (equivalence classes of) weakly measurable, essentially bounded, $\mathscr{L}(\mathscr{E})$-valued functions on $\mathbf{T}$ will be denoted by $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$. The space $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$ is a C^* algebra with the algebraic operations defined pointwise, norm defined by the equation

$$\| A(\cdot)\| = \operatorname*{ess-sup}_{z\in\mathbf{T}} \| A(z)\|, \qquad A(\cdot) \in \mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})},$$

and involution defined by the equation

$$A^*(z) = (A(z))^*, \qquad A(\cdot) \in \mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}.$$

The space $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ is defined to be the set of all those functions $A(\cdot)$ in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$ with the property that the Fourier coefficient of $A(\cdot)$ of negative index vanishes. The space $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ is a closed subalgebra of $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$ and every function in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ admits a natural analytic continuation into Δ.

The algebra $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$ may be represented as an algebra of operators on $\mathbf{L}^2_{\mathscr{E}}$ in the following way. Let $A(\cdot)$ be in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$; then the operator A (without parentheses) determined by $A(\cdot)$ is defined by the equation

$$(Af)(z) = A(z)f(z), \qquad f(\cdot) \in \mathbf{L}^2_{\mathscr{E}}. \tag{2.1}$$

If $A(\cdot)$ is in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$, then the operator on $\mathbf{L}^2_{\mathscr{E}}$ determined by $A(\cdot)$ (via (2.1)) leaves $\mathbf{H}^2_{\mathscr{E}}$ invariant if and only if $A(\cdot)$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$.

We pause to emphasize the following notational conventions. A letter followed by parentheses with a dot between them will always denote a function. Functions in $\mathbf{L}^2_{\mathscr{E}}$, $\mathbf{H}^2_{\mathscr{E}}$, or in certain spaces which we will introduce later will usually be denoted by small Roman or Greek letters. If we wish to consider such functions simply as vectors in these spaces we will often drop the parentheses. Functions in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$ will always be denoted by capital Roman or Greek letters. If $A(\cdot)$ is in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$, then A (without parentheses) will always denote the operator in $\mathbf{L}^2_{\mathscr{E}}$ determined by $A(\cdot)$ via (2.1). If $A(\cdot)$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$, then the restriction of A to $\mathbf{H}^2_{\mathscr{E}}$ will always be denoted by A_+. The only exceptions to these conventions are as follows. When convenient, we will identify vectors in $\mathscr{E}$ with the constant functions in $\mathbf{L}^2_{\mathscr{E}}$. An operator T on $\mathscr{E}$ will be identified with the constant function in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$ which is almost everywhere equal to T. The same symbol, T, will then be used to denote any of the various roles that it might play, i.e., an operator on $\mathscr{E}$, a function in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$, an operator on $\mathbf{L}^2_{\mathscr{E}}$, or an operator on $\mathbf{H}^2_{\mathscr{E}}$. In the special situations where these identifications are encountered, we will point them out and no confusion should arise.

The function $U(\cdot)$ in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$ is defined by the equation $U(z) = zI$

where I is the identity operator on $\mathscr{E}$. The operator U on $\mathbf{L}^2_{\mathscr{E}}$ is known as the *bilateral shift* of multiplicity equal to the dimension of $\mathscr{E}$. The operator U leaves $\mathbf{H}^2_{\mathscr{E}}$ invariant and U_+ is known as the *unilateral shift* of multiplicity equal to the dimension of $\mathscr{E}$. The fundamental relationships between the operators U, U_+ and the algebras $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$ and $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ may be expressed as follows. The correspondence which sends a function $A(\cdot)$ in $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$ to the operator A on $\mathbf{L}^2_{\mathscr{E}}$ is a $*$-isomorphism of $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$ onto the commutant of U (see Ref. [3, Chap. II, Section 2, Theorem 1]), while the correspondence which sends a function $A(\cdot)$ in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ to the operator A_+ on $\mathbf{H}^2_{\mathscr{E}}$ is an isometric isomorphism of $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ onto the commutant of U_+ (see Ref. [18, p. 183]).

The importance of shifts in the theory which surrounds our main theorem is that they form the building blocks from which universal models for operators can be constructed. We make this statement precise in Theorem I (below) which is a slightly restricted form of a more general result due to Sz.-Nagy and Foiaş, and is central to their structure theory of operators.

We say that a contraction T belongs to class C_{00} provided the powers of T and of T^* converge strongly to zero. A function $\Theta(\cdot)$ in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ is called an inner function in case $\Theta(z)$ is unitary a.e.

THEOREM I. [18, Chap. 6, No. 2]. *Let T be contraction on a Hilbert space $\mathscr{N}$ and assume that* T *is class* C_{00}. *Then there is a Hilbert space $\mathscr{E}$ and an inner function $\Theta(\cdot)$ in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ such that T is unitarily equivalent to*

$$PU_+ \mid \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}, \tag{2.2}$$

where P denotes the projection of $\mathbf{H}^2_{\mathscr{E}}$ onto $\mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}$. Moreover, if T is unitarily equivalent to

$$P'U_+ \mid \mathbf{H}^2_{\mathscr{F}} \ominus \Theta_+'\mathbf{H}^2_{\mathscr{F}}, \tag{2.2$'$}$$

where $\mathscr{F}$ is another Hilbert space, $\Theta'(\cdot)$ is an inner function in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{F})}$, and P' is the projection of $\mathbf{H}^2_{\mathscr{F}}$ onto $\mathbf{H}^2_{\mathscr{F}} \ominus \Theta_+'\mathbf{H}^2_{\mathscr{F}}$, then there exist two (not necessarily distinct) isometric isomorphisms η and τ from $\mathscr{E}$ onto $\mathscr{F}$ such that $\tau^{-1}\Theta(z)\eta = \Theta'(z)$ a.e. Conversely, any operator written in the form (2.2) *with arbitrary inner function is a contraction of class C_{00}.*

The essentially unique inner function of Theorem I determined by a contraction T in class C_{00} has a concrete representation in terms of T, but, since we won't need this representation until Section 7, we postpone a discussion of it until then.

In Ref. [19] Sz.-Nagy and Foiaş characterized the commutant of a contraction written in the form (2.2). We state their characterization in the following theorem.

THEOREM II. *Let $\mathscr{E}$ be a Hilbert space, let $\Theta(\cdot)$ be an inner function in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$, let $\mathscr{H} = \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}$, and let $T = PU_+ \mid \mathscr{H}$, where P is the projection of $\mathbf{H}^2_{\mathscr{E}}$ onto $\mathscr{H}$. An operator S on $\mathscr{H}$ commutes with T if and only if there exists a function $\Phi(\cdot)$ in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ such that*

$$\Phi_+\Theta_+\mathbf{H}^2_{\mathscr{E}} \subset \Theta_+\mathbf{H}^2_{\mathscr{E}} \qquad \text{and} \qquad S = P\Phi_+ \mid \mathscr{H}.$$

Moreover, given S, one can choose $\Phi(\cdot)$ so that $\| \Phi(\cdot)\| = \| S \|$.

Let $C^{\infty}(\mathscr{E})$ denote the space of compact operators on $\mathscr{E}$. The space $C^{\infty}(\mathscr{E})$ is the only proper norm-closed two-sided ideal in $\mathscr{L}(\mathscr{E})$ when $\mathscr{E}$ is infinite-dimensional. When $\mathscr{E}$ is finite-dimensional, $\mathscr{L}(\mathscr{E}) = C^{\infty}(\mathscr{E})$. We let $\mathbf{C}_{C^{\infty}(\mathscr{E})}$ denote the space of all $C^{\infty}(\mathscr{E})$-valued functions on $\mathbf{T}$ which are continuous with respect to the norm topology.

With the preliminaries out of the way, we may state our theorem, which characterizes compact operators in the commutant of an operator written in the form (2.2).

THEOREM III. *Let $\mathscr{E}$ be a Hilbert space, let $\Theta(\cdot)$ be an inner function in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$, let $\mathscr{H} = \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}$, and let $T = PU_+ \mid \mathscr{H}$, where P is the projection of $\mathbf{H}^2_{\mathscr{E}}$ onto $\mathscr{H}$. Let S be an operator in $\mathscr{L}(\mathscr{H})$ commuting with T and suppose $S = P\Phi_+ \mid \mathscr{H}$ where $\Phi(\cdot)$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ and $\Phi_+\Theta_+\mathbf{H}^2_{\mathscr{E}} \subseteq \Theta_+\mathbf{H}^2_{\mathscr{E}}$ (see Theorem* II). *Then S is compact if and only if $\Theta^*(\cdot)\,\Phi(\cdot)$ is in the space $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^{\infty}(\mathscr{E})}$.*

In view of Theorems I and II, Theorem III characterizes the compact operators in the commutant of the most general contraction in class C_{00}. Observe, also that if $\mathscr{E}$ is one-dimensional, then Theorem III is Sarason's Theorem 2. Sarason's proof makes use of the duality properties of $H^{\infty} + C$ where H^{∞} is the space of bounded analytic functions on Δ and C is the space of continuous complex-valued functions on $\mathbf{T}$. In Sections 4 and 5, we establish analogous properties of $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^{\infty}(\mathscr{E})}$. Once these properties are established we show, in Section 6, that Sarason's proof can be modified to give a proof of Theorem III.

3. In the context of the Sz.-Nagy–Foiaş structure theory of operators, one does not necessarily wish to restrict one's attention to contractions of class C_{00}. It is natural to ask, therefore, if there is an

extension of Theorem III to arbitrary contractions. In a sense to be made clear presently, the answer to such a question is no.

A contraction T is said to be *completely nonunitary* (c.n.u.) in case there is no nonzero reducing subspace for T on which T is unitary. Every contraction splits into a direct sum of a unitary operator and a c.n.u. contraction [18, Chap. I, Theorem 3.2]. For most purposes, therefore, when studying the structural properties of contractions, it suffices to restrict one's attention to c.n.u. contractions. In fact, it is almost always assumed in the Sz.-Nagy–Foiaş structure theory that the contractions studied are c.n.u.

The following proposition establishes the generality of Theorem III.

PROPOSITION 3.1. *If T is a c.n.u. contraction on a Hilbert space, and if there is a compact operator K with zero kernel and dense range commuting with T, then T is of class C_{00}.*

The proof of Proposition 3.1 is contained in the following two lemmas whose proofs we omit. The first is a paraphrase of a theorem by Deckard, Douglas, and Pearcy [1] and the second is due to Foguel [6].

LEMMA 3.2. *Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be a bounded net of operators on a Hilbert space converging to an operator T in the weak operator topology. If there exists a compact operator K with zero kernel and dense range such that T_λ commutes with K for every λ, then the net $\{T_\lambda\}_{\lambda\in\Lambda}$ converges to T in the strong operator topology and the net $\{T_\lambda{}^*\}_{\lambda\in\Lambda}$ converges to T^* in the strong operator topology.*

LEMMA 3.3. *If T is a c.n.u. contraction on a Hilbert space, then the powers of T and of T^* converge to zero in the weak operator topology.*

We observe in passing that the hypothesis in Proposition 3.1 that K have zero kernel and dense range is, from certain points of view, not particularly restrictive; in fact, it is natural. For example, if one wants to study the invariant subspace properties of an operator T with a compact operator K in its commutant then the hypothesis that K have zero kernel and dense range is desirable. The reason for this is that both the closure of the range of K and the kernel of K are invariant under T. In general, without additional information, nothing much can be said about the restriction of T to the kernel of K and, likewise, nothing much can be said about the behavior of T off the range of K. Therefore, to obtain interesting information, one might as well assume at the outset that the range of K is dense and that its kernel is zero.

4. In this section we gather together for reference a number of facts about compact operators which we will need in the sequel. The results stated here without proof may be found in either Ref. [17] or Ref. [5, Vol. II, Chap. XI].

Let $\mathscr{E}$ be a fixed Hilbert space. For the most part, we will assume that the dimension of $\mathscr{E}$ is infinite since everything we will say is true (most likely, vacuously true) when the dimension of $\mathscr{E}$ is finite. For each p, $1 \leqslant p < \infty$, there is an ideal of operators contained in $C^\infty(\mathscr{E})$ which we denote by $C^p(\mathscr{E})$ and which is defined as follows. An operator T belongs to $C^p(\mathscr{E})$ if and only if T is in $C^\infty(\mathscr{E})$ and the sequence of eigenvalues of $(T^*T)^{1/2}$ lies in l^p. If T is in $C^p(\mathscr{E})$ then the $C^p(\mathscr{E})$ norm of T is defined to be the l^p norm of the sequence of eigenvalues of $(T^*T)^{1/2}$. With respect to this norm, $C^p(\mathscr{E})$ is a Banach algebra. Moreover, if T is in $C^p(\mathscr{E})$ and X and Y are in $\mathscr{L}(\mathscr{E})$, then XTY is in $C^p(\mathscr{E})$ and

$$\| XTY \|_{C^p(\mathscr{E})} \leqslant \| X \|_{\mathscr{L}(\mathscr{E})} \| T \|_{C^p(\mathscr{E})} \| Y \|_{\mathscr{L}(\mathscr{E})} .$$

Remark 4.1. If $\mathscr{E}$ is finite-dimensional, then for each p, $1 \leqslant p \leqslant \infty$, $C^p(\mathscr{E})$ and $\mathscr{L}(\mathscr{E})$ are the same finite-dimensional vector space and, consequently, any two locally convex topologies on this space are homeomorphic. Therefore, although the norms on these spaces are different, in general, they are equivalent. As we will see, the difference in norms is important when studying the duality properties of these spaces.

We will need the following proposition. Although minor variants of it appear in the literature, we do not know of a reference for the proposition as we have stated it. Consequently, we will indicate a proof.

PROPOSITION 4.2. *Let T be in $C^p(\mathscr{E})$ and let $\{P_n\}_{n-1}^\infty$ be an increasing sequence of projections converging to the identity on $\mathscr{E}$ in the strong operator topology. Then* $\lim_{n\to\infty} \| P_nTP_n - T \|_{C^p(\mathscr{E})} = 0$.

Indication of Proof. If T has finite rank the result is an easy computation. The proposition, therefore, follows from the observation that the operators of finite rank are dense in $C^p(\mathscr{E})$ with respect to the $C^p(\mathscr{E})$ norm [5, Vol. II, Chap. XI, Section 9, Lemma 11].

A simple consequence of Proposition 4.2 is the following corollary.

COROLLARY 4.3. *For each* p, $1 \leqslant p \leqslant \infty$, *the Banach space* $C^p(\mathscr{E})$ *is separable.*

The ideals $C^1(\mathscr{E})$ and $C^2(\mathscr{E})$ are of particular importance and are called the space of trace class operators and Hilbert–Schmidt operators, respectively. If T is in $C^1(\mathscr{E})$ we define the trace of T, denoted by $\mathrm{Tr}(T)$, to be the sum $\sum_{i=1}^{\infty} (Te_i, e_i)$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathscr{E}$. The series is absolutely convergent and the the sum is independent of the choice of basis [5, Vol. II, p. 1097]. The trace satisfies

(a) $|\mathrm{Tr}(T)| \leqslant \| T \|_{C^1(\mathscr{E})}$, and
(b) If X is in $\mathscr{L}(\mathscr{E})$ and T is in $C^1(\mathscr{E})$, then $\mathrm{Tr}(XT) = \mathrm{Tr}(TX)$.

With respect to the inner product defined by the equation

$$(A, B) = \mathrm{Tr}(B^*A), \qquad A, B \in C^2(\mathscr{E}),$$

$C^2(\mathscr{E})$ is a (separable) Hilbert space and $\| A \|_{C^2(\mathscr{E})}^2 = (A, A)$ for every A in $C^2(\mathscr{E})$. In addition, this inner product satisfies the following important property. Let A and B be in $C^2(\mathscr{E})$ and let D be in $\mathscr{L}(\mathscr{E})$; then

$$(A, DB^*) = \mathrm{Tr}(D^*AB) = (B, A^*D).$$

The dual of $C^\infty(\mathscr{E})$ is $C^1(\mathscr{E})$ where the pairing is given by the equation $\langle A, B\rangle = \mathrm{Tr}(AB)$, $A \in C^\infty(\mathscr{E})$, $B \in C^1(\mathscr{E})$. The dual of $C^1(\mathscr{E})$ is $\mathscr{L}(\mathscr{E})$, and the pairing is expressed similarly; that is, $\langle A, B\rangle = \mathrm{Tr}(AB)$, $A \in C^1(\mathscr{E})$, $B \in \mathscr{L}(\mathscr{E})$ [17, Chap. 4].

5. In this section we investigate the duality properties of the space $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$. These properties play the same role in the proof of Theorem III as the analogous properties of $H^\infty + C$ play in Sarason's Theorem 2. The remaining technical lemmas needed for the proof of Theorem III will also be presented here.

Throughout this section, E will denote an arbitrary complex Banach space and $\mathbf{L}_E^p$ $(1 \leqslant p < \infty)$ will denote the space of all measurable E-valued functions $f(\cdot)$ defined on $\mathbf{T}$ such that $\|f(\cdot)\|^p$ is integrable with respect to m. The space $\mathbf{L}_E^p$ is a Banach space with its norm defined by the equation

$$\| f(\cdot)\|_{\mathbf{L}_E^p}^p = \int_{\mathbf{T}} \| f(z)\|_E^p \, dm, \qquad f(\cdot) \in \mathbf{L}_E^p .$$

(For the general properties of vector-valued measures and integrals, we refer the reader to the book by Dinculeanu [2].) Let $\mathbf{H}_E^1$ be the space of all those functions $f(\cdot)$ in $\mathbf{L}_E^1$ with the property that the Fourier coefficients of $f(\cdot)$ with negative index vanish. The functions in $\mathbf{H}_E^1$ admit a natural analytic continuation into Δ and we let $\mathbf{H}_{0E}^1$ denote the subspace of $\mathbf{H}_E^1$ consisting of those functions which vanish at the origin. The following proposition is an immediate application of Theorem 8 and 9 of Ref. [2, Chap II, Section 13] and the fact that the Banach space $C^1(\mathscr{E})$ is separable (see Cor. 4.3).

PROPOSITION 5.1. *The dual space* $\mathbf{L}_{C^1(\mathscr{E})}^1$ *is* $\mathbf{L}_{\mathscr{L}(\mathscr{E})}^\infty$ *and the pairing is given by the equation*

$$\langle A(\cdot), B(\cdot)\rangle = \int_{\mathbf{T}} \operatorname{Tr}(A(z)\,B(z))\,dm,$$

$$A(\cdot) \in \mathbf{L}_{C^1(\mathscr{E})}^1, \qquad B(\cdot) \in \mathbf{L}_{\mathscr{L}(\mathscr{E})}^\infty .$$

COROLLARY 5.2. *The annihilator of* $\mathbf{H}_{0C^1(\mathscr{E})}^1$ *in* $\mathbf{L}_{\mathscr{L}(\mathscr{E})}^\infty$ *is* $\mathbf{H}_{\mathscr{L}(\mathscr{E})}^\infty$.

Corollary 5.2 and a standard application of the Hahn–Banach Theorem yield the following proposition.

PROPOSITION 5.3. *The dual space of* $\mathbf{H}_{0C^1(\mathscr{E})}^1$ *is* $\mathbf{L}_{\mathscr{L}(\mathscr{E})}^\infty/\mathbf{H}_{\mathscr{L}(\mathscr{E})}^\infty$.

Let $\mathbf{C}_E$ denote the space of functions from $\mathbf{T}$ into E which are continuous with respect to the norm topology on E. Then with respect to pointwise algebraic operations and norm defined by the equation

$$\| f(\cdot)\|_{\mathbf{C}_E} = \sup_{z\in\mathbf{T}} \| f(z)\|_E , \qquad f(\cdot) \in \mathbf{C}_E ,$$

$\mathbf{C}_E$ is a Banach space. In a fashion analogous to the classical Riesz Representation Theorem, the dual space of $\mathbf{C}_E$ may be identified with the space of regular Borel, E^*-valued measures defined on $\mathbf{T}$ (see Ref. [2, p. 337]); we denote this space by $\mathbf{M}_{E^*}$. Let $\mathbf{A}_E$ denote the subspace of $\mathbf{C}_E$ consisting of those functions whose Fourier coefficients of negative index vanish. In Ref. [15] Ryan proved a vector-valued F. and M. Riesz Theorem implying that in the case when E^* is separable, the annihilator of $\mathbf{A}_E$ in $\mathbf{M}_{E^*}$ is isometrically isomorphic to $\mathbf{H}_{0E^*}^1$. (See Ref. [13, Chap. II] for an alternative proof of Ryan's theorem used here.) Since the dual space of $C_{(\mathscr{E})}^\infty$ is $C^1(\mathscr{E})$ and $C^1(\mathscr{E})$ is separable (see Corollary 4.3), we may apply Ryan's result and a well-known consequence of the Hahn–Banach Theorem to obtain the following proposition.

PROPOSITION 5.4. *The dual space of* $\mathbf{C}_{C^\infty(\mathscr{E})}/\mathbf{A}_{C^\infty(\mathscr{E})}$ is $\mathbf{H}^1_{0C^1(\mathscr{E})}$.

Propositions 5.3 and 5.4 together imply that the space $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}/\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ is the second dual of $\mathbf{C}_{C^\infty(\mathscr{E})}/\mathbf{A}_{C^\infty(\mathscr{E})}$. The natural imbedding J of $\mathbf{C}_{C^\infty(\mathscr{E})}/\mathbf{A}_{C^\infty(\mathscr{E})}$ into $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}/\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ carries a coset of the form $K(\cdot) + \mathbf{A}_{C^\infty(\mathscr{E})}$, where $K(\cdot)$ is in $\mathbf{C}_{C^\infty(\mathscr{E})}$, to the coset $K(\cdot) + \mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$. If $\mathscr{P}$ denotes the quotient map of $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$ onto $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}/\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$, then

$$\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})} = \mathscr{P}^{-1}J[\mathbf{C}_{C^\infty(\mathscr{E})}/\mathbf{A}_{C^\infty(\mathscr{E})}]. \tag{5.1}$$

Equation (5.1) establishes the duality properties of $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$ and yields the following proposition.

PROPOSITION 5.5. *The linear manifold* $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$ *is closed in* $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$.

We point out that when $\mathscr{E}$ is infinite-dimensional there is another operator-valued analog of $H^\infty + C$ beside $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$; it is the space $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{\mathscr{L}(\mathscr{E})}$. This space, too, is closed in $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$; but, since we don't need this fact in the sequel, we refer the reader to Ref. [13] for a proof.

A trigonometric E-valued polynomial is a sum of the form $\sum_{k=-n}^{n} a_k z^k$ where a_k is in E and z ranges over $\mathbf{T}$.

LEMMA 5.6. *The space of trigonometric E-valued polynomials is dense in* $\mathbf{C}_E$.

Proof. If $A(\cdot)$ is in $\mathbf{C}_E$, then the arithmetic means of the Fourier series of $A(\cdot)$ converge to $A(\cdot)$ in the norm on $\mathbf{C}_E$. To see this, simply replace absolute values by norms in the scalar-valued analog of this result presented in Ref. [10, p. 17].

PROPOSITION 5.7. *The space* $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$ *is a subalgebra of* $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$.

Proof. By Lemma 5.6 and Proposition 5.5, it is the closure of the subalgebra of $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$ consisting of functions of the form $\Omega(\cdot) + K(\cdot)$ where $\Omega(\cdot)$ is in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ and $K(\cdot)$ is a $C^\infty(\mathscr{E})$-valued trigonometric polynomial.

We observe in passing that the space $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{\mathscr{L}(\mathscr{E})}$ is also a subalgebra of $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$; it is the closure of the algebra whose elements are of the form $\Omega(\cdot) + K(\cdot)$ where $\Omega(\cdot)$ is in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ and $K(\cdot)$ is an $\mathscr{L}(\mathscr{E})$-valued trigonometric polynomial.

The following proposition will be of use to us in the proof of Theorem III.

PROPOSITION 5.8. *Let $\{P_n\}_{n=1}^{\infty}$ be an increasing sequence of projections which converges to the identity operator on $\mathscr{E}$ in the strong operator topology.*

(a) *Let $f(\cdot)$ be a function in $\mathbf{L}^p_{C^q(\mathscr{E})}$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, and let $f_n(z) = P_n f(z) P_n$ for all z in $\mathbf{T}$. Then*

$$\lim_{n\to\infty} \|f_n(\cdot) - f(\cdot)\|_{\mathbf{L}^p_{C^q(\mathscr{E})}} = 0.$$

(b) *Let $f(\cdot)$ be a function in $\mathbf{C}_{C^q(\mathscr{E})}$, $1 \leqslant q \leqslant \infty$, and let $f_n(z) = P_n f(z) P_n$ for all z in $\mathbf{T}$. Then*

$$\lim_{n\to\infty} \|f_n(\cdot) - f(\cdot)\|_{\mathbf{C}_{C^q(\mathscr{E})}} = 0.$$

Proof. (a) By Proposition 4.2, $\lim_{n\to\infty} \|f_n(z) - f(z)\|_{C^q(\mathscr{E})} = 0$ for each z in $\mathbf{T}$; also, $\|f_n(z) - f(z)\|_{C^q(\mathscr{E})} \leqslant 2\,\|f(z)\|_{C^q(\mathscr{E})}$. By Lebesgue's dominated convergence theorem, then, we see that

$$\begin{aligned} &\lim_{n\to\infty} \|f_n(\cdot) - f(\cdot)\|^p_{\mathbf{L}^p_{C^q(\mathscr{E})}} \\ &\quad = \lim_{n\to\infty} \int_{\mathbf{T}} \|f_n(z) - f(z)\|^p_{C^q(\mathscr{E})}\, dm \\ &\quad = 0. \end{aligned}$$

(b) Given $\epsilon > 0$, choose $C^q(\mathscr{E})$-valued trigonometric polynomial $Q(\cdot)$ such that $\|f(\cdot) - Q(\cdot)\| < \epsilon/3$ (see Lemma 5.6. Since $Q(\cdot)$ is a trigonometric polynomial with coefficients in $C^q(\mathscr{E})$, it is possible to find an n_0 such that $\| P_n Q(\cdot) P_n - Q(\cdot)\|_{\mathbf{C}_{C^q(\mathscr{E})}} < \epsilon/3$ whenever $n > n_0$ (see Proposition 4.2). We find, therefore, that if $n > n_0$,

$$\begin{aligned} \|f_n(\cdot) - f(\cdot)\|_{\mathbf{C}_{C^q(\mathscr{E})}} &\leqslant \|f_n(\cdot) - P_n Q(\cdot) P_n\|_{\mathbf{C}_{C^q(\mathscr{E})}} \\ &\quad + \| P_n Q(\cdot) P_n - Q(\cdot)\|_{\mathbf{C}_{C^q(\mathscr{E})}} + \|Q(\cdot) - f(\cdot)\|_{\mathbf{C}_{C^q(\mathscr{E})}} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

A celebrated theorem of F. Riesz states that if a function $f(\cdot)$ is in H^1, then it is possible to factor $f(\cdot)$ as the product of two functions in H^2. In Ref. [16, Theorem 4], Sarason proved a far-reaching generalization of this result which is basic to our considerations. (See Ref. [19, Section 3] for a slightly modified form and proof of this theorem by Sarason.) Actually, we will use a modification of Sarason's result. We will state his theorem as Theorem IV′, and the form in which we will use it as Theorem IV.

THEOREM IV′. *Let* $A(\cdot)$ *be a function in* $\mathbf{H}^1_{C^1(\mathscr{E})}$; *then* $A(\cdot)$ *can be factored as* $A(\cdot) = A_1(\cdot)\,A_2(\cdot)$ *where each function* $A_i(\cdot)$ *is in* $\mathbf{H}^2_{C^2(\mathscr{E})}$, $i = 1, 2$, $A_2(z)^*\,A_2(z) = (A(z)^*\,A(z))^{1/2}$, *and* $A_2(z)\,A_2(z)^* = A_1(z)^*\,A_1(z)$ *a.e.*

THEOREM IV. *Let* $A(\cdot)$ *be a function in* $\mathbf{H}^1_{0C^1(\mathscr{E})}$. *Then* $A(\cdot)$ *can be factored as* $A(\cdot) = A_1(\cdot)\,A_2(\cdot)$ *where* $A_1(\cdot)$ *is in* $\mathbf{H}^2_{C^2(\mathscr{E})}$, $A_2(\cdot)$ *is in* $\mathbf{H}^2_{0C^2(\mathscr{E})}$, *and* $\| A_i(\cdot)\|^2_{\mathbf{H}^2_{C^2(\mathscr{E})}} = \| A(\cdot)\|_{\mathbf{H}^1_{C^1(\mathscr{E})}}$, $i = 1, 2$.

To obtain Theorem IV from Theorem IV′, simply observe that a function $A(\cdot)$ is in $\mathbf{H}^1_{0C^1(\mathscr{E})}$ if and only if the function $B(\cdot) = U^*(\cdot)\,A(\cdot)$ is in $\mathbf{H}^1_{C^1(\mathscr{E})}$. If $B(\cdot)$ is factored as $B(\cdot) = B_1(\cdot)\,B_2(\cdot)$ where the functions $B_i(\cdot)$ satisfy the conditions of Theorem IV′, then, letting $A_1(\cdot) = B_1(\cdot)$ and $A_2(\cdot) = U(\cdot)\,B_2(\cdot)$, $A(\cdot) = A_1(\cdot)\,A_2(\cdot)$ and the functions $A_i(\cdot)$ satisfy the conclusion of Theorem IV′.

6. The objective of this section is the proof of Theorem III. Before presenting the proof we make some remarks and introduce some notation. We will assume throughout the proof that the Hilbert space $\mathscr{E}$ is infinite-dimensional for, as we shall see, the proof for the case when the dimension of $\mathscr{E}$ is finite is actually contained in that for the case when the dimension of $\mathscr{E}$ is infinite. We will let $\{e_j\}_{j=1}^{\infty}$ be a fixed orthonormal basis for $\mathscr{E}$, we will let $\mathscr{E}_n$ denote the span of the first n basis vectors, and we will denote by Q_n the projection of $\mathscr{E}$ onto $\mathscr{E}_n$. The projections Q_n are each of finite rank and the sequence $\{Q_n\}_{n=1}^{\infty}$ converges to the identity on $\mathscr{E}$ in the strong operator topology. When it is convenient, we will regard the projection P of $\mathbf{H}^2_{\mathscr{E}}$ onto $\mathscr{H}$ (in the statement of Theorem III) as the projection of $\mathbf{L}^2_{\mathscr{E}}$ onto $\mathscr{H}$. Observe that if $F(\cdot)$ is in $\mathbf{H}^2_{C^2(\mathscr{E})}$, then the $\mathscr{E}$-valued function obtained by applying $F(\cdot)$ to a vector e in $\mathscr{E}$ pointwise is in $\mathbf{H}^2_{\mathscr{E}}$. We will denote this function either by $F(\cdot)e$ or Fe. In context, this notation will cause no confusion. Finally, we will identify the spaces $\mathbf{L}^2_{\mathscr{E}_n}$ and $\mathbf{H}^2_{\mathscr{E}_n}$ as subspaces of $\mathbf{L}^2_{\mathscr{E}}$ and $\mathbf{H}^2_{\mathscr{E}}$ (respectively) in the obvious way.

Proof of the Sufficiency. Suppose $\Theta^*(\cdot)\,\Phi(\cdot) = \Omega(\cdot) + K(\cdot)$, where $\Omega(\cdot)$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ and $K(\cdot)$ is in $\mathbf{C}_{C^{\infty}(\mathscr{E})}$; then $\Phi(\cdot) = \Theta(\cdot)(\Omega(\cdot) + K(\cdot))$. We wish to show that $S = P\Phi_+ \mid \mathscr{H}$ is compact. It is implicit in the proof of Proposition 5.8 that we may find a subsequence $\{\mathscr{E}_{n(k)}\}_{k=1}^{\infty}$ of the sequence $\{\mathscr{E}_n\}_{n=1}^{\infty}$ and a sequence of trigonometric polynomials $\{K_k(\cdot)\}_{k=1}^{\infty}$ such that (a) $\lim_{k\to\infty} \| K_k(\cdot) - K(\cdot)\|_{\mathbf{C}_{C^{\infty}(\mathscr{E})}} = 0$ and (b) the ranges of the coefficients of the polynomial $K_k(\cdot)$ are contained in $\mathscr{E}_{n(k)}$. (Simply approximate $K(\cdot)$ by arithmetic means of the Fourier series of $K(\cdot)$ and then truncate each mean with a suitable Q_n.)

If we let $\Phi_k(\cdot) = \Theta(\cdot)(\Omega(\cdot) + K_k(\cdot))$, then the sequence $\{\Phi_k(\cdot)\}_{k=1}^{\infty}$ converges to $\Phi(\cdot)$ in the norm on $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E})}$, and therefore, the sequence of operators $\{P\Phi_k \mid \mathscr{H}\}_{k=1}^{\infty}$ converges to $S = P\Phi_+ \mid \mathscr{H}$ in the norm on $\mathscr{L}(\mathscr{H})$. To show that S is compact, it suffices, then, to prove that each of the operators $P\Phi_k \mid \mathscr{H}$ is of finite rank.

To this end, write $K_k(\cdot) = K_k^+(\cdot) + K_k^-(\cdot)$, where $K_k^+(\cdot)$ is the analytic part of $K_k(\cdot)$ and $K_k^-(\cdot)$ is the conjugate analytic part of K_k; that is, $K_k^+(\cdot)$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ and $(K_k^-)^*(\cdot)$ is in $\mathbf{H}^{\infty}_{0\mathscr{L}(\mathscr{E})}$. Since $(\Omega(\cdot) + K_k^+(\cdot))\,\mathbf{H}^2_{\mathscr{E}} \subset \mathbf{H}^2_{\mathscr{E}}$, we obtain the equation $P\Theta K_k^- \mid \mathscr{H} = P\Theta[(\Omega + K_k^+) + K_k^-] \mid \mathscr{H} = P\Phi_k \mid \mathscr{H}$. Let d_k be the degree of K_k^-. We then see that

$$\begin{aligned} K_k^-\mathbf{H}^2_{\mathscr{E}} &\subset U^{*d_k}\mathbf{H}^2_{\mathscr{E}_{n(k)}} \\ &= U^{*d_k}\mathscr{E}_{n(k)} \oplus U^{*d_k-1}\mathscr{E}_{n(k)} \oplus \cdots \oplus U^*\mathscr{E}_{n(k)} \oplus \mathbf{H}^2_{\mathscr{E}_{n(k)}} \\ &= \mathscr{T} \oplus \mathbf{H}^2_{\mathscr{E}_{n(k)}}\,. \end{aligned}$$

Observe that the space $\mathscr{T}$ is a finite-dimensional subspace of $\mathbf{L}^2_{\mathscr{E}}$. The operator Θ on $\mathbf{L}^2_E$ is unitary since $\Theta(\cdot)$ is an inner function, and therefore we may write

$$\Theta K_k^-\mathbf{H}^2_{\mathscr{E}} \subset \Theta\mathscr{T} \oplus \Theta\mathbf{H}^2_{\mathscr{E}_{n(k)}}$$

and $\Theta\mathscr{T}$, too, is a finite-dimensional subspace of $\mathbf{L}^2_{\mathscr{E}}$. Finally, we have $[P\Theta K_k^-]\mathscr{H} \subset [P\Theta K_k^-]\,\mathbf{H}^2_{\mathscr{E}} \subset [P\Theta]\mathscr{T}$. Since this last space is a finite-dimensional subspace of $\mathscr{H}$, the proof that $P\Theta_k \mid \mathscr{H}$ $(= P\Theta K_k^- \mid \mathscr{H})$ if of finite rank is complete. Hence, the sufficiency of our condition is proved.

Proof of the Necessity. Suppose $S = P\Phi_+ \mid \mathscr{H}$ is compact; we wish to show that $\Theta(\cdot)\,\Phi(\cdot)$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^{\infty}(\mathscr{E})}$. Let λ be the linear functional on $\mathbf{H}^1_{0C^1(\mathscr{E})}$ defined by the equation

$$\lambda(F(\cdot)) = \int_{\mathbf{T}} \mathrm{Tr}(\Theta^*(z)\Phi(z)F(z))\,dm, \qquad F(\cdot) \in \mathbf{H}^1_{0C^1(\mathscr{E})}\,.$$

We will show that λ is continuous with respect to the weak* topology on $\mathbf{H}^1_{0C^1(\mathscr{E})}$. By the duality properties of $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^{\infty}(\mathscr{E})}$ developed in Section 5, this will show that $\Theta(\cdot)\,\Phi(\cdot)$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^{\infty}(\mathscr{E})}$. To show that λ is weak* continuous, it suffices to show that the kernel of λ is weak* sequentially closed [5, Vol. I, Theorem I on p. 426 and Theorem 7 on p. 429].

To this end, let $\{F_k(\cdot)\}_{k=1}^{\infty}$ by a sequence in the kernel of λ converging weak* to a function $F(\cdot)$ in $\mathbf{H}^1_{0C^1(\mathscr{E})}$. For each positive integer n we write

$$\begin{aligned}
&\lambda(F_k(\cdot)) - \lambda(F(\cdot)) \\
&= \left\{\int_{\mathbf{T}} \mathrm{Tr}(\Theta^*(z)\Phi(z)F_k(z))\, dm - \int_{\mathbf{T}} \mathrm{Tr}(Q_n\Theta^*(z)\Phi(z)Q_nF_k(z)\, dm\right\} \\
&\quad + \left\{\int_{\mathbf{T}} \mathrm{Tr}(Q_n\Theta^*(z)\Phi(z)Q_nF_k(z))\, dm - \int_{\mathbf{T}} \mathrm{Tr}(Q_n\Theta^*(z)\Phi(z)Q_nF(z))\, dm\right\} \\
&\quad + \left\{\int_{\mathbf{T}} \mathrm{Tr}(Q_n\Theta^*(z)\Phi(z)Q_nF(z))\, dm - \int_{\mathbf{T}} \mathrm{Tr}(\Theta^*(z)\Phi(z)F(z))\, dm\right\} \\
&= I_{kn} + J_{kn} + L_n .
\end{aligned}$$

We will prove the following:

$$\lim_{n\to\infty} I_{kn} = 0 \quad \text{uniformly in } k; \tag{6.A}$$

$$\text{For each fixed } n, \quad \lim_{k\to\infty} J_{kn} = 0; \tag{6.B}$$

$$\lim_{n\to\infty} L_n = 0. \tag{6.C}$$

It is clear that the statements (6.A), (6.B), and (6.C) together imply that $\lim_{k\to\infty} \lambda(F_k(\cdot)) = \lambda(F(\cdot))$, and, since $\lambda(F_k(\cdot)) = 0$ for all k, we will obtain $\lambda(F(\cdot)) = 0$, which is what we wish to prove.

The following lemma is fundamental to the proof of the first two of our three assertions and is an operator-valued analog of Lemma 2.1 of Ref. [16].

LEMMA 6.1. *Let $\mathscr{E}$ be a separable Hilbert space, let $\Theta(\cdot)$ be an inner function in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$, let $\mathscr{H} = \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}$, and let P denote the projection of $\mathbf{L}^2_{\mathscr{E}}$ onto $\mathscr{H}$. Suppose $\Phi(\cdot)$ is in $\mathscr{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ and satisfies the relation $\Phi_+(\Theta_+\mathbf{H}^2_{\mathscr{E}}) \subset \Theta_+\mathbf{H}^2_{\mathscr{E}}$. Then if $f(\cdot)$ is in $\mathbf{H}^2_{\mathscr{E}}$ and $g(\cdot)$ is in $\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}}$, we have $(\Theta^*\Phi f, g)_{\mathbf{L}^2_{\mathscr{E}}} = (P\Phi Pf, \Theta g)_{\mathbf{L}^2_{\mathscr{E}}}$.*

Proof. Clearly,

$$(\Theta^*\Phi f, g)_{\mathbf{L}^2_{\mathscr{E}}} = (\Phi f, \Theta g)_{\mathbf{L}^2_{\mathscr{E}}} . \tag{6.1}$$

Since g is in $\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}}$, and since Θ is a unitary operator on $\mathbf{L}^2_{\mathscr{E}}$, we see that Θg is in $\mathbf{L}^2_{\mathscr{E}} \ominus \Theta\mathbf{H}^2_{\mathscr{E}} = \mathscr{H} + (\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}})$. We may therefore write

$$(\Phi f, \Theta g)_{\mathbf{L}^2_{\mathscr{E}}} = (\Phi f, P\Theta g)_{\mathbf{L}^2_{\mathscr{E}}} , \tag{6.2}$$

since Φf is in $\mathbf{H}^2_{\mathscr{E}}$. Since $f - Pf$ is in $\Theta\mathbf{H}^2_{\mathscr{E}}$, $\Phi(f - Pf)$ is in $\Theta\mathbf{H}^2_{\mathscr{E}}$ by the hypothesis on Φ. Thus we may write

$$\begin{aligned}(\Phi f, P\Theta g)_{\mathbf{L}^2_{\mathscr{E}}} &= (\Phi[Pf + (1-P)f], P\Theta g)_{\mathbf{L}^2_{\mathscr{E}}} \\ &= (\Phi Pf, P\Theta g)_{\mathbf{L}^2_{\mathscr{E}}} + ((1-P)\,\Phi(1-P)f, P\Theta g)_{\mathbf{L}^2_{\mathscr{E}}} \qquad (6.3)\\ &= (\Phi Pf, P\Theta g)_{\mathbf{L}^2_{\mathscr{E}}} = (P\Theta Pf, \Theta g)_{\mathbf{L}^2_{\mathscr{E}}}.\end{aligned}$$

The proof is completed by combining Eqs. (6.1), (6.2), and (6.3).

Proof of (6.A). Using Theorem IV, we write $F_k(\cdot) = F_{1k}(\cdot)F_{2k}(\cdot)$, where $F_{1k}(\cdot)$ is in $\mathbf{H}^2_{C^2(\mathscr{E})}$, $F_{2k}(\cdot)$ is in $\mathbf{H}^2_{0C^2(\mathscr{E})}$ and $\| F_{ik}(\cdot)\|^2_{\mathbf{H}^2_{C^2(\mathscr{E})}} = \| F_k(\cdot)\|_{\mathbf{H}^1{}_{C^1(\mathscr{E})}}$, $i = 1, 2$. Since the sequence $\{F_k(\cdot)\}_{k=1}^{\infty}$ converges in the weak* topology on $\mathbf{H}^1_{0C^1(\mathscr{E})}$, it is bounded. Consequently, the sequences $\{F_{ik}(\cdot)\}_{k=1}^{\infty}$, $i = 1, 2$, are bounded in the $\mathbf{H}_{C^2(\mathscr{E})}$ norm. Let M be a common bound for the $\mathbf{H}^2_{C^2(\mathscr{E})}$ norms of the $F_{ik}(\cdot)$, $i = 1, 2$. We now write a lengthy sequence of inequalities. The justification for each will be given after they are written.

$$\begin{aligned}
| I_{kn} | &= \Big| \int_{\mathbf{T}} \operatorname{Tr}(\Theta^*(z)\Phi(z)F_k(z))\, dm \\
&\qquad - \int_{\mathbf{T}} \operatorname{Tr}(Q_n\Theta^*(z)\Phi(z)Q_nF_k(z))\, dm \Big| && (6.A.1)\\
&= \Big| \int_{\mathbf{T}} (\Theta^*(z)\Phi(z)F_{1k}(z), F_{2k}^*(z))_{C^2(\mathscr{E})}\, dm \\
&\qquad - \int_{\mathbf{T}} (Q_n\Theta^*(z)\Phi(z)Q_nF_{1k}(z), F_{2k}^*(z))_{C^2(\mathscr{E})}\, dm \Big| && (6.A.2)\\
&= \Big| \int_{\mathbf{T}} \sum_j (\Theta^*(z)\Phi(z)F_{1k}(z)e_j, F_{2k}^*(z)e_j)\, dm \\
&\qquad - \int_{\mathbf{T}} \sum_j (Q_n\Theta^*(z)\Phi(z)Q_nF_{1k}(z)e_j, F_{2k}^*(z)\, e_j) dm \Big| && (6.A.3)\\
&= \Big| \sum_j (\Theta^*\Phi F_{1k}e_j, F_{2k}^*e_j)_{\mathbf{L}^2_{\mathscr{E}}} - \sum_j (Q_n\Theta^*\Phi Q_nF_{1k}e_j, F_{2k}^*e_j)_{\mathbf{L}^2_{\mathscr{E}}} \Big| && (6.A.4)\\
&= \Big| \sum_j ([\Theta^*P\Phi P - Q_n\Theta^*P\Phi PQ_n]F_{1k}e_j, F_{2k}^*e_j)_{\mathbf{L}^2_{\mathscr{E}}} \Big| && (6.A.5)\\
&\leqslant \| \Theta^*P\Phi P - Q_n\Theta^*P\Phi PQ_n \|_{\mathscr{L}(\mathbf{L}^2_{\mathscr{E}})} \\
&\qquad \times \sum_j \| F_{1k}e_j \|_{\mathbf{L}^2_{\mathscr{E}}} \| F_{2k}^*e_j \|_{\mathbf{L}^2_{\mathscr{E}}} && (6.A.6)
\end{aligned}$$

$$\leqslant \| \Theta^* P\Phi P - Q_n \Theta^* P\Phi P Q_n \|_{\mathscr{L}(\mathbf{L}^2_{\mathscr{E}})}$$

$$\times \left(\sum_j \| F_{1k} e_j \|^2_{\mathbf{L}^2_{\mathscr{E}}}\right)^{1/2} \left(\sum_j \| F^*_{2k} e_j \|^2_{\mathbf{L}^2_{\mathscr{E}}}\right)^{1/2} \tag{6.A.7}$$

$$\leqslant \| \Theta^* P\Phi P - Q_n \Theta^* P\Phi P Q_n \|_{\mathscr{L}(\mathbf{L}^2_{\mathscr{E}})} M^2. \tag{6.A.8}$$

The transitions from (6.A.1) to (6.A.2) and from (6.A.2) to (6.A.3) are based on the fact that $F_k(\cdot) = F_{1k}(\cdot) F_{2k}(\cdot)$ and the basic properties of the inner product on $C^2(\mathscr{E})$ outlined in Section 4. To pass from (6.A.3) to (6.A.4) one simply needs to interchange the order of summation and integration, and this is possible since the series in question are absolutely convergent by the Cauchy–Schwartz inequality. The passage from (6.A.4) to (6.A.5) is an application of Lemma 6.1 using the facts that for all j and k, $F_{1k}(\cdot)\, e_j$ is in $\mathbf{H}^2_{\mathscr{E}}$ (since $F_{1k}(\cdot)$ is in $\mathbf{H}^2_{C^2(\mathscr{E})}$) and $F^*_{2k}(\cdot)\, e_j$ is in $\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}}$ (since $F_{2k}(\cdot)$ is in $\mathbf{H}^2_{0C^2(\mathscr{E})}$). The transition from (6.A.5) to (6.A.6) is clear and that from (6.A.6) to (6.A.7) is an application of the Cauchy–Schwartz inequality. Finally, to pass from (6.A.7) to (6.A.8) observe that

$$\left(\sum_j \| F_{1k} e_j \|^2_{\mathbf{L}^2_{\mathscr{E}}}\right)^{1/2} = \| F_{1k} \|_{\mathbf{L}^2_{C^2(\mathscr{E})}},$$

$$\left(\sum_j \| F^*_{2k} e_j \|^2_{\mathbf{L}^2_{\mathscr{E}}}\right)^{1/2} = \| F^*_{2k} \|_{\mathbf{L}^2_{C^2(\mathscr{E})}} = \| F_{2k} \|_{\mathbf{L}^2_{C^2(\mathscr{E})}},$$

and that the $\mathbf{L}^2_{C^2(\mathscr{E})}$ norms of these sequences are bounded by M.

Since $P\Phi_+ \mid \mathscr{H}$ is compact, $P\Phi P$ is a compact operator on $\mathbf{L}^2_{\mathscr{E}}$; therefore, $\Theta^* P\Phi P$ is compact also. Consequently, since the projections Q_n, regarded as operators on $\mathbf{L}^2_{\mathscr{E}}$, converge strongly to the identity on $\mathbf{L}^2_{\mathscr{E}}$, the sequence

$$\{\| \Theta^* P\Phi P - Q_n \Theta^* P\Phi P Q_n \|_{\mathscr{L}(\mathbf{L}^2_{\mathscr{E}})}\}_{n=1}^{\infty}$$

converges to zero independently of k (see Proposition 4.2). Thus $\lim_{n\to\infty} I_{kn} = 0$ uniformly in k and the proof of (6.A) is complete.

Proof of (6.B). To prove (6.B) it suffices to show that for each n, $Q_n \Theta^*(\cdot)\, \Phi(\cdot)\, Q_n$ is in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$, and to show this it suffices to show that $Q_n \Theta^*(\cdot)\, \Phi(\cdot)\, Q_n$ regarded as an element of $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E}_n)}$ is in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E}_n)} + \mathbf{C}_{C^\infty(\mathscr{E}_n)}$. We proceed just as we did in the beginning of the proof of Theorem III. Let λ_n be the linear functional on $\mathbf{H}^1_{0C^1(\mathscr{E}_n)}$ defined by the equation

$$\lambda_n(G(\cdot)) = \int_{\mathbf{T}} \mathrm{Tr}(Q_n \Theta^*(z) \Phi(z) Q_n G(z))\, dm,$$

where $G(\cdot)$ is in $\mathbf{H}^1_{0C^1(\mathscr{E}_n)}$. We wish to show that the kernel of λ_n is weak* sequentially closed.

To this end, let $\{G_k(\cdot)\}_{k=1}^{\infty}$ be a sequence in the kernel of λ_n converging weak* to $G(\cdot)$ in $\mathbf{H}^1_{0C^1(\mathscr{E}_n)}$. Applying Theorem IV, we write $G_k(\cdot) = G_{1k}(\cdot)\, G_{2k}(\cdot)$, where $G_{1k}(\cdot)$ is in $\mathbf{H}^2_{C^2(\mathscr{E}_k)}$, $G_{2k}(\cdot)$ is in $\mathbf{H}^2_{0C^2(\mathscr{E}_n)}$, and

$$\| G_{ik}(\cdot)\|^2_{\mathbf{H}^2_{C^2(\mathscr{E}_n)}} = \| G_k(\cdot)\|_{\mathbf{H}^1_{C^1(\mathscr{E}_n)}} \qquad i = 1, 2.$$

Since the $G_k(\cdot)$ are bounded in the $\mathbf{H}^1_{0C^1(\mathscr{E}_n)}$ norm, the $G_{ik}(\cdot)$ are bounded in the $\mathbf{H}^2_{0C^2(\mathscr{E}_n)}$ norm. Consequently, by passing to a subsequence, if necessary, we may assume that the $G_{ik}(\cdot)$ converge to functions $G_i(\cdot)$, $i = 1, 2$, in the weak topology on the Hilbert space $\mathbf{H}^2_{C^2(\mathscr{E}_n)}$. We show, first, that $G(\cdot) = G_1(\cdot)\, G_2(\cdot)$.

Since the sequence $\{G_{ik}(\cdot)\}_{k=1}^{\infty}$ converges to $G_i(\cdot)$, $i = 1, 2$, in the weak topology on $\mathbf{H}^2_{C^2(\mathscr{E}_n)}$, it follows that for each w, $|w| < 1$, the sequence of operators $\{G_{ik}(w)\}_{k=1}^{\infty}$ converges to $G_i(w)$ in the weak topology on $C^2(\mathscr{E}_n)$, $i = 1, 2$. Since $\mathscr{E}_n$ is a finite-dimensional Hilbert space, so is $C^2(\mathscr{E}_n)$, and consequently, on $C^2(\mathscr{E}_n)$ the weak and strong topologies coincide (see Remark 4.1). Hence we may conclude that for each w, $|w| < 1$, and for each $i = 1, 2$, $\lim_{k\to\infty} \| G_{ik}(w) - G_i(w)\|_{C^2(\mathscr{E}_n)} = 0$. We remark in passing that what we have shown so far also implies $G_2(\cdot)$ is in $\mathbf{H}^2_{0C^2(\mathscr{E}_n)}$. This will be important later. Also, since $\mathscr{E}_n$ is finite-dimensional, multiplication is continuous with respect to any locally convex topology on $\mathscr{L}(\mathscr{E}_n)$. Therefore we see that for each w, $|w| < 1$, $\lim_{k\to\infty} \| G_{1k}(w)\, G_{2k}(w) - G_1(w)\, G_2(w)\|_{C^1(\mathscr{E}_n)} = 0$. On the other hand, since the sequence $\{G_k(\cdot)\}_{k=1}^{\infty}$ converges to $G(\cdot)$ in the weak* topology on $\mathbf{H}^1_{0C^1(\mathscr{E}_n)}$, we may conclude that for each w, $|w| < 1$, the sequence $\{G_k(w)\}_{k=1}^{\infty}$ converges to $G(w)$ in the weak* topology on $C^1(\mathscr{E}_n)$. But since $\mathscr{E}_n$ is finite-dimensional we obtain $\lim_{k\to\infty} \| G_k(w) - G(w)\|_{C^1(\mathscr{E}_n)} = 0$. In sum, we have shown that for each w, $|w| < 1$, $G(w) = \lim_{k\to\infty} G_k(w) = \lim_{k\to\infty} G_{1k}(w)\, G_{2k}(w) = G_1(w)\, G_2(w)$ where the limits are taken with respect to the norm on $C^1(\mathscr{E}_n)$. Thus $G(\cdot) = G_1(\cdot)\, G_2(\cdot)$, as we wished to prove.

Employing reasoning similar to that in the proof of (6.A), we write

$$\begin{aligned}
&\lambda_n(G_k(\cdot)) - \lambda_n(G(\cdot)) \\
&\quad = \int_{\mathbf{T}} \operatorname{Tr}(Q_n\Theta^*(z)\Phi(z)Q_nG_k(z))\, dm - \int_{\mathbf{T}} \operatorname{Tr}(Q_n\Theta^*(z)\Phi(z)Q_nG(z))\, dm \\
&\quad = \int_{\mathbf{T}} (Q_n\Theta^*(z)\Phi(z)Q_nG_{1k}(z),\, G^*_{2k}(z))_{C^2(\mathscr{E}_n)}\, dm \\
&\qquad - \int_{\mathbf{T}} (Q_n\Theta^*(z)\Phi(z)Q_nG_1(z),\, G_2{}^*(z))_{C^2(\mathscr{E}_n)}\, dm
\end{aligned}$$

$$= \sum_{j=1}^{n} (Q_n\Theta^*\Phi Q_n G_{1k}e_j\,,\, G_{2k}^*e_j)_{\mathbf{L}^2_{\mathscr{E}_n}} - \sum_{j=1}^{n} (Q_n\Theta^*\Phi Q_n G_1 e_j\,,\, G_2{}^*e_j)_{\mathbf{L}^2_{\mathscr{E}_n}}\,. \tag{6.B.1}$$

Regarding $\mathscr{E}_n$ as a subspace of $\mathscr{E}$, we can write (6.B.1)

$$= \sum_{j=1}^{n} (Q_n\Theta^*\Phi Q_n G_{1k}e_j\,,\, G_{2k}^*e_j)_{\mathbf{L}^2_{\mathscr{E}}} - \sum_{j=1}^{n} (Q_n\Theta^*\Phi Q_n G_1 e_j\,,\, G_2{}^*e_j)_{\mathbf{L}^2_{\mathscr{E}}} \tag{6.B.2}$$

$$= \sum_{j=1}^{n} (Q_n\Theta^*P\Phi P Q_n G_{1k}e_j\,,\, G_{2k}^*e_j)_{\mathbf{L}^2_{\mathscr{E}}}$$

$$- \sum_{j=1}^{n} (Q_n\Theta^*P\Phi P Q_n G_1 e_j\,,\, G_2{}^*e_j)_{\mathbf{L}^2_{\mathscr{E}}}\,. \tag{6.B.3}$$

The passage from (6.B.2) to (6.B.3) is justified by Lemma 6.1 and the fact that for each k and $j = 1, 2,..., n$, $G_{1k}(\cdot)\, e_j$ and $G_1(\cdot)\, e_j$ are in $\mathbf{H}^2_{\mathscr{E}}$ and $G_{2k}^*(\cdot)\, e_j$ and $G_2{}^*(\cdot)\, e_j$ are in $\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}}$. (This last statement is valid since the $G_{2k}(\cdot)$ and $G_2(\cdot)$ are in $\mathbf{H}^2_{0C^2(\mathscr{E}_n)}$.) Furthermore, since the sequence $\{G_{ik}(\cdot)\}_{k=1}^{\infty}$ converges to $G_i(\cdot)$, $i = 1, 2$, in the weak topology on $\mathbf{H}^2_{C^2(\mathscr{E}_n)}$, we see that for each j, $j = 1, 2,..., n$, the sequences of vectors $\{G_{1k}(\cdot)\, e_j\}_{k=1}^{\infty}$ and $\{G_{2k}^*(\cdot)\, e_j\}_{k=1}^{\infty}$ converge to $G_1(\cdot)\, e_j$ and $G_2{}^*(\cdot)\, e_j$, respectively, in the weak topology on $\mathbf{L}^2_{\mathscr{E}}$. Since $P\Phi_+ \mid \mathscr{H}$ is a compact operator on $\mathscr{H}$, $P\Phi P$ is a compact operator on $\mathbf{L}^2_{\mathscr{E}}$, and consequently, $Q_n\Theta^*P\Phi P Q_n$ is also a compact operator on $\mathbf{L}^2_{\mathscr{E}}$. It follows, therefore, that for each j, $j = 1, 2,..., n$, the sequence of vectors $\{Q_n\Theta^*P\Phi P Q_n G_{1k}e_j\}_{k=1}^{\infty}$ converges strongly to $Q_n\Theta^*P\Phi P Q_n G_1 e_j$, and we may conclude that the sequence in (6.B.3) converges to zero. Thus we have shown that $0 = \lim_{k\to\infty} \lambda_n(G_k) = \lambda_n(G)$, and therefore, the kernel of λ_n is weak* sequentially closed. Hence, $Q_n\Theta^*(\cdot)\,\Phi(\cdot)\, Q_n$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$ and the proof of (6.B) is complete.

Proof of (6.C). To prove (6.C), observe that

$$|\, L_n \,| = \left| \int_{\mathbf{T}} \operatorname{Tr}(\Theta^*(z)\Phi(z)\{Q_n F(z) Q_n - F(z)\}\, dm \right|$$

$$\leqslant \|\, \Theta^*(\cdot)\Phi(\cdot)\|_{\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}} \int_{\mathbf{T}} \| Q_n F(z) Q_n - F(z)\|_{C^1(\mathscr{E})}\, dm.$$

Since the sequence of integrals $\{\int_{\mathbf{T}} \| Q_n F(z)\, Q_n - F(z)\|_{C^1(\mathscr{E})}\, dm\}_{n=1}^{\infty}$ converges to zero by Proposition 5.8, we see that $\lim_{n\to\infty} L_n = 0$, and (6.C) is proved. This completes the proof of Theorem III.

7. In this section we present two immediate consequences of Theorem III.

Let T be a completely nonunitary compact contraction on a separable Hilbert space. Since T is compact, its nonzero spectrum consists solely of a discrete set of eigenvalues, and since a c.n.u. contraction can have no eigenvalues of modulus one (see Ref. [18, p. 75]), the spectral radius of T is strictly less than one. We see, therefore, that T is of class C_{00} (in fact, the powers of T and of T^* converge to zero in norm). Thus, by Theorem I, T is unitarily equivalent to an operator of the form

$$PU_+ \mid \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}, \tag{7.1}$$

where $\mathscr{E}$ is an auxiliary (separable space, $\Theta(\cdot)$ is an inner function, and P is the projection of $\mathbf{H}^2_{\mathscr{E}}$ onto $\mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}$. In Section 2 we mentioned that $\Theta(\cdot)$ has a specific representation in terms of T. We digress momentarily to describe this representation.

Let T be an arbitrary c.n.u. contraction on a Hilbert space $\mathscr{H}$. Let $D = (I - T^*T)^{1/2}$, $D_* = (I - TT^*)^{1/2}$ (where I denotes the identity operator on $\mathscr{H}$), and let $\mathscr{D}$ and $\mathscr{D}_*$ be the closures of the ranges of D and D_*, respectively. Consider the function $\Theta_T(\cdot)$ defined by the equation

$$\Theta_T(z) = -T + zD_*(I - zT^*)^{-1}D, \qquad |z| < 1. \tag{7.2}$$

It can be shown that for each z, $|z| < 1$, $\Theta_T(z)\mathscr{D} \subset \mathscr{D}_*$. Regarded as an operator from $\mathscr{D}$ into $\mathscr{D}_*$, $\Theta_T(z)$ has norm less than or equal to one for $|z| < 1$; and the function $\Theta_T(\cdot)$ is analytic in the open unit disk. Sz.-Nagy and Foiaş call $\Theta_T(\cdot)$ the *characteristic operator function* associated with T and we refer the reader to Ref. [18, Chap VI] for a complete discussion of its properties. They show there that a necessary and sufficient condition that T be in class C_{00} is that $\mathscr{D}$ and $\mathscr{D}_*$ have the same dimension and that $\Theta_T(\cdot)$ have boundary values on $\mathbf{T}$ which are unitary operators mapping $\mathscr{D}$ into $\mathscr{D}_*$. In this case they show that T is unitarily equivalent to $PU_+ \mid \mathbf{H}^2_{\mathscr{D}_*} \ominus \Theta_{T+}\mathbf{H}^2_{\mathscr{D}}$ where P is the projection of $\mathbf{H}^2_{\mathscr{D}_*}$ onto $\mathbf{H}^2_{\mathscr{D}_*} \ominus \Theta_{T+}\mathbf{H}^2_{\mathscr{D}}$. If $\mathscr{D}$ and $\mathscr{D}_*$ are identified with $\mathscr{E}$, then $\Theta_T(\cdot)$ is an inner function and we have the representation in (7.1).

Let T be a c.n.u. compact contraction. Since the spectral radius of T is strictly less than one, it follows easily from Eq. (7.2) that $\Theta_T(\cdot)$ is analytic in a neighborhood of the closed unit disk. Also, the spectral theorem implies that $D = 1 + C$ and $D_* = 1 + C_*$ where C and C_* are compact operators. Consequently, if we expand $\Theta_T(\cdot)$ in a power series about the origin, we see that $\Theta_T(\cdot)$ can be written as

$$\Theta_T(z) = z + K(z),$$

where $K(z)$ is analytic in a neighborhood of the closed unit disk and has compact values for each z in its domain. Specifically,

$$\begin{aligned}\Theta_T(z) &= -T + \sum_{n=1}^{\infty} z^n D_* T^{*n-1} D \\ &= -T + z(I + C_*)(I + C) + \sum_{n=2}^{\infty} z^n (I + C_*) T^{*n-1}(I + C) \\ &= z + \Big[-T + z(C_* + C + C_*C) + \sum_{n=2}^{\infty} z^n (I + C_*) T^{*n-1}(I + C)\Big] \\ &= z + K(z).\end{aligned}$$

The quantity in square brackets (which is $K(z)$) is a compact operator for each z in some neighborhood of the closed unit disk, since each term in the series is compact and the series converges absolutely and uniformly in a neighborhood of the closed unit disk. We have thus proved one half of the following theorem.

THEOREM V. *Let T be a c.n.u. contraction on a separable Hilbert space. Then T is compact if and only if the characteristic operator function $\Theta_T(\cdot)$ for T is inner and can be written*

$$\Theta_T(z) = z + K(z),$$

where $K(z)$ is analytic in a neighborhood of the closed unit disk and has compact values for each z in its domain.

Proof. The necessity of the condition has already been demonstrated. Suppose that $\Theta_T(\cdot)$ is inner and is expressible as $z + K(z)$, where $K(z)$ satisfies the conditions of the theorem. Then T is unitarily equivalent to $PU_+ \mid \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_{T+}\mathbf{H}^2_{\mathscr{E}}$, where $\mathscr{D}$ and $\mathscr{D}_*$ have been identified with $\mathscr{E}$ as above. From the form of $\Theta_T(\cdot)$ we see that for each z in $\mathbf{T}$, $\Theta_T{}^*(z)\, U(z) = \Theta_T(z)^* z = 1 + z(K(z))^*$, and we may conclude that $\Theta_T{}^*(\cdot)\, U(\cdot)$ is in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$. By Theorem III, T is compact.

It may seem that Theorem V can be improved. Observe that in the proof of the sufficiency of the condition in Theorem V it is sufficient to assume that $K(\cdot)$ belongs to $A_{C^\infty(\mathscr{E})}$. We point out, however, that the assumptions (a) that $\Theta_T(z) = z + K(z)$ is an inner function and (b) that $K(\cdot)$ is in $\mathbf{A}_{C^\infty(\mathscr{E})}$ together imply that $K(\cdot)$ is analytic in a neighborhood of the closed unit disk. This fact follows from an operator-valued analog of Schwartz's reflection principal which states that if a function $\Psi(\cdot)$ in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ is continuous on an open subarc α of $\mathbf{T}$ and if $\Psi(z)$ is unitary for all z in α, then $\Psi(\cdot)$ can be continued analytically across α. For a proof of this fact, see Ref. [9] (see also Ref. [19, p. 214]).

DEFINITION 7.1. A family of contractions $\{T(s)\}_{s\geqslant 0}$ on a Hilbert space $\mathscr{H}$ is called a *semigroup of contractions of class* C_{00} in case

(a) For each $s, t \leqslant 0$, $T(s+t) = T(s)\,T(t)$;
(b) $T(0) = I$, the identity on $\mathscr{H}$;
(c) For each $s_0 \leqslant 0$, $\lim_{s\to s_0} T(s) = T(s_0)$;
(d) $\lim_{s\to\infty} T(s) = \lim_{s\to 0} T(s)^* = 0$.

(The limits in (c) and (d) are taken with respect to the strong operator topology on $\mathscr{L}(\mathscr{H})$.)

Let $\{T(s)\}_{s\geqslant 0}$ be a semigroup of contractions of class C_{00}. For such a semigroup, Sz.-Nagy and Foiaş [18, Chap. 3, No. 8] have shown that there exists a contraction T called the *cogenerator* of $\{T(s)\}_{s\geqslant 0}$ such that T is in class C_{00} and such that

$$T(s) = \exp\left\{s\left(\frac{T+1}{T-1}\right)\right\}, \qquad s \geqslant 0. \tag{7.3}$$

The right side of Eq. (7.3) is defined using their functional calculus for contractions (see Ref. [18, Chap. 3]). We know that T is unitarily equivalent to

$$PU_+ \mid \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_{T+}\mathbf{H}^2_{\mathscr{E}}, \tag{7.4}$$

where $\Theta_T(\cdot)$ is the characteristic operator function for T and $\mathscr{E}$ is identified with $\mathscr{D}$ and $\mathscr{D}_*$. Under the unitary equivalence which sends T to the operator in (7.4), $T(s)$ is sent to the operator

$$P\varphi_{s+} \mid \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_{T+}\mathbf{H}^2_{\mathscr{E}}, \qquad s \geqslant 0,$$

where $\varphi_s(\cdot)$ is the function defined by the equation

$$\varphi_s(z) = \exp\left\{s\left(\frac{z+1}{z-1}\right)\right\}, \qquad |z| < 1. \tag{7.5}$$

THEOREM VI. *Let $\mathscr{E}$ be a separable Hilbert space, let $\Theta(\cdot)$ be an inner function in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$, and let $\mathscr{H} = \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}$. If $\{T(s)\}_{s\geqslant 0}$ is the semigroup of contractions on $\mathscr{H}$ defined by the equation*

$$T(s) = P\varphi_{s+} \mid \mathscr{H}, \qquad s \geqslant 0,$$

where P is the projection of $\mathbf{H}^2_{\mathscr{E}}$ onto $\mathscr{H}$ and $\varphi_s(\cdot)$ is the function defined by Eq. (7.5), *then $T(s_0)$ is compact for some $s_0 > 0$ if and only if $\Theta^*(\cdot)\, \varphi_{s_0}(\cdot)$ is in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$.*

Proof. The proof follows at once from Theorem III.

Theorem V may have applications to scattering theory, which we explain. The characteristic operator function $\Theta_T(\cdot)$ for the cogenerator of a semigroup of contractions of class C_{00}, $\{T(s)\}_{s\geq 0}$, is closely related to the scattering operator associated with the semigroup (see Refs. [11, Chap. 2] and [18, p. 268]). In scattering theory it seems to be important to know when there is an $s_0 > 0$ such that $T(s_0)$ is compact. Our theorem establishes a necessary and sufficient condition that $T(s_0)$ be compact in terms of the characteristic operator function of the cogenerator (or, equivalently, in terms of the scattering operator for the semigroup). Whether or not this condition will prove to be useful, we do not know.

8. In this section we define a class of operators which we will call generalized Hankel operators and show how to modify the proof of Theorem III in order to determine a necessary and sufficient condition that a generalized Hankel operator be compact. This condition, which is a generalization of a theorem due to Hartman [7], will be given in Theorem VII (below). Since an alternate proof of Theorem VII has recently been obtained by Page [14] independently from our investigations, we will only outline our proof and refer the reader to our dissertation [13] for complete details. Also, in this section we will prove a theorem (Theorem VIII, below) which relates operators in the commutant of a contraction to a class of generalized Hankel operators. Using this result and Theorem VII we give an alternate proof of Theorem III.

Let R denote the operator on $\mathbf{L}^2_{\mathscr{E}}$ defined by the equation $(Rf)(z) = f(\bar{z})$, $f(\cdot) \in \mathbf{L}^2_{\mathscr{E}}$, and observe that R is a unitary operator on $\mathbf{L}^2_{\mathscr{E}}$ satisfying the equation $R^2 = I$, where I is the identity operator on $\mathbf{L}^2_{\mathscr{E}}$.

DEFINITION 8.1. Let $\Phi(\cdot)$ be in $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$. The *generalized Hankel* operator H_Φ on $\mathbf{H}^2_{\mathscr{E}}$ determined by $\Phi(\cdot)$ is defined by the equation

$$H_\Phi = P_1 R \Phi \mid \mathbf{H}^2_{\mathscr{E}},$$

where P_1 denotes the projection of $\mathbf{L}^2_{\mathscr{E}}$ into $\mathbf{H}^2_{\mathscr{E}}$.

In Ref. [14] Page showed that the generalized Hankel operators are precisely the solutions X of the equation

$$XU_+ = U_+^* X, \tag{8.1}$$

where U_+ is the unilateral shift on $\mathbf{H}^2_{\mathscr{E}}$; that is, for a given X satisfying (8.1) there exists a $\Phi(\cdot)$ in $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}$ such that $X = H_\Phi$. The function $\Phi(\cdot)$ determined by X is not unique; however, it is unique modulo $\mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$ (those functions in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})}$ which vanish at the origin). In fact, we have the following proposition whose proof may be found in either Ref. [14] or Ref. [13, p. 90].

PROPOSITION 7.1. *Let $\mathscr{H}_{\mathscr{E}}$ denote the space of Hankel operators on $\mathbf{H}^2_{\mathscr{E}}$ and let Γ be the map from the Banach space $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}/\mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$ to $\mathscr{H}_{\mathscr{E}}$ defined by the equation $\Gamma(\Phi(\cdot) + \mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}) = H_\Phi$. Then Γ is an isometric isomorphism of $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}/\mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$ onto $\mathscr{H}_{\mathscr{E}}$ (endowed with the operator norm).*

A rephrasing of Propositions 5.3 and 5.4 (as well as their proofs) shows that $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}/\mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$ is the dual space of $\mathbf{H}^1_{C^1(\mathscr{E})}$, which is the dual space of $\mathbf{C}_{C^\infty(\mathscr{E})}/\mathbf{A}_{0C^\infty(\mathscr{E})}$ (where $\mathbf{A}_{0C^\infty(\mathscr{E})}$ is the subspace of $\mathbf{A}_{C^\infty(\mathscr{E})}$ consisting of those functions which vanish at the origin). Thus Proposition 8.1 allows us to identify $\mathscr{H}_{\mathscr{E}}$ as a second dual space. As we will see in Theorem VII, the compact generalized Hankel operators are precisely those which lie in the image of the natural imbedding into $\mathscr{H}_{\mathscr{E}}$ of the predual of the predual of $\mathscr{H}_{\mathscr{E}}$. We first need the following lemma, of interest in its own right, which describes a relation between a linear functional on $\mathbf{H}^1_{C^1(\mathscr{E})}$ and the generalized Hankel operator which it determines via Proposition 8.1.

LEMMA 8.2. *Let λ be a linear functional on $\mathbf{H}^1_{C^1(\mathscr{E})}$ and let $\Phi(\cdot)+\mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$ be the coset corresponding to λ under the identification of the dual space of $\mathbf{H}^1_{C^1(\mathscr{E})}$ with $\mathbf{L}^\infty_{\mathscr{L}(\mathscr{E})}/\mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$. Let $F(\cdot)$ be in $\mathbf{H}^1_{C^1(\mathscr{E})}$, let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis for $\mathscr{E}$, and let $F(\cdot) = F_1(\cdot)\,F_2(\cdot)$ ba a factoring of $F(\cdot)$ into the product of two functions in $\mathbf{H}^2_{C^2(\mathscr{E})}$ (see Theorem IV′). Then*

$$\lambda(F) = \sum_{j=1}^{\infty} (H_\Phi F_1 e_j\,, \tilde{F}_2 e_j),$$

where $\tilde{F}_2(\cdot)$ is that function whose value at each point z is $F_2(\bar{z})^*$.

Proof. The proof is a computation:

$$\begin{aligned}\lambda(F(\cdot)) &= \int_{\mathbf{T}} \operatorname{Tr}(\Phi(z)F(z))\,dm \\ &= \int_{\mathbf{T}} \operatorname{Tr}(\Phi(z)F_1(z)F_2(z))\,dm \\ &= \int_{\mathbf{T}} (\Phi(z)F_1(z), F_2{}^*(z))_{C^2(\mathscr{E})}\,dm\end{aligned}$$

$$= \int_{\mathbf{T}} \sum_{j=1}^{\infty} (\Phi(z)F_1(z)e_j\,, F_2{}^*(z)e_j)\, dm$$

$$= \sum_{j=1}^{\infty} \int_{\mathbf{T}} (\Phi(z)F_1(z)e_j\,, R\tilde{F}_2(z)e_j)\, dm$$

$$= \sum_{j=1}^{\infty} \int_{\mathbf{T}} (P_1R\Phi(z)F_1(z)e_j\,, \tilde{F}_2(z)e_j)\, dm$$

$$= \sum_{j=1}^{\infty} (H_\Phi F_1 e_j\,, \tilde{F}_2 e_j).$$

(Note that we may interchange summation and integration in the above equation, since the series in question is absolutely convergent a.e.)

THEOREM VII. *The generalized Hankel operator* H_Φ *is compact if and only if* $\Phi(\cdot)$ *lies in a coset of the form* $K(\cdot) + \mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$, *where* $K(\cdot)$ *is in* $\mathbf{C}_{C^\infty(\mathscr{E})}$.

Proof. Suppose $\Phi(\cdot) = \Omega(\cdot) + K(\cdot)$, where $\Omega(\cdot)$ is in $\mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$ and $K(\cdot)$ is in $\mathbf{C}_{C^\infty(\mathscr{E})}$. As in the proof of Theorem III, we can find a sequence of trigonometric polynomials $\{K_k(\cdot)\}_{k=1}^\infty$ whose coefficients are of finite rank and which satisfy $\lim_{k\to\infty} \| K(\cdot) - K_k(\cdot)\|_{\mathbf{C}_{C^\infty(\mathscr{E})}} = 0$. If we let $\Phi_k(\cdot) = K_k(\cdot) + \Omega(\cdot)$, then by Proposition 8.1,

$$\lim_{k\to\infty} \| H_{\Phi_k} - H_\Phi \| = 0.$$

An estimation on the ranges of the H_{Φ_k} similar to those made in the proof of Theorem III shows that each H_{Φ_k} is of finite rank. It follows, therefore, that H_Φ is compact.

Conversely, suppose H_Φ is compact. In order to show that $\Phi(\cdot)$ lies in a coset of the form $K(\cdot) + \mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})}$, where $K(\cdot)$ is in $\mathbf{C}_{C^\infty(\mathscr{E})}$, it suffices to show that the linear functional λ on $\mathbf{H}^1_{C^1(\mathscr{E})}$ determined by $\Phi(\cdot)$ is continuous with respect to the weak* topology, and, to do this, it suffices to show, as in the proof of Theorem III, that the kernel of λ is weak* sequentially closed. To this end, let $\{F_k(\cdot)\}_{k=1}^\infty$ be a sequence in the kernel of λ converging weak* to a function $F(\cdot)$ in $\mathbf{H}^1_{C^1(\mathscr{E})}$. Also, let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis for $\mathscr{E}$, let $\mathscr{E}_n$ denote the span of the first n e_j's, and let Q_n denote the projection of $\mathscr{E}$ onto $\mathscr{E}_n$. For each positive integer we write

$$\begin{aligned}
&\lambda(F_k(\cdot)) - \lambda(F(\cdot)) \\
&\quad = \left\{\int_{\mathbf{T}} \operatorname{Tr}(\Phi(z)F_k(z))\, dm - \int_{\mathbf{T}} \operatorname{Tr}(Q_n\Phi(z)Q_nF_k(z))\, dm\right\} \\
&\qquad + \left\{\int_{\mathbf{T}} \operatorname{Tr}(Q_n\Phi(z)Q_nF_k(z))\, dm - \int_{\mathbf{T}} \operatorname{Tr}(Q_n\Phi(z)Q_nF(z))\, dm\right\} \\
&\qquad + \left\{\int_{\mathbf{T}} \operatorname{Tr}(Q_n\Phi(z)Q_nF(z))\, dm - \int_{\mathbf{T}} \operatorname{Tr}(\Phi(z)F(z))\, dm\right\} \\
&\quad = I_{nk} + J_{nk} + L_n .
\end{aligned}$$

The following assertions together clearly imply that $\lim_{k\to\infty} \lambda(F_k(\cdot)) = \lambda(F(\cdot))$. Since $\lambda(F_k(\cdot)) = 0$ for all k, $\lambda(F(\cdot)) = 0$, and this is what we wish to prove. The assertions are

$$\lim_{n\to\infty} I_{nk} = 0 \quad \text{uniformly in } k; \tag{8.A}$$

$$\lim_{k\to\infty} J_{nk} = 0 \quad \text{for each fixed } n; \tag{8.B}$$

$$\lim_{n\to\infty} L_n = 0. \tag{8.C}$$

To prove these three statements, we first observe that

$$H_{Q_n\Phi Q_n} = Q_nH_\Phi Q_n .$$

Secondly, we note that since $\lim_{k\to\infty} F_k(\cdot) = F(\cdot)$ in the weak* topology, the $\mathbf{H}^1_{C^1(\mathscr{E})}$ norms of the $F_k(\cdot)$ and $F(\cdot)$ are bounded by some positive constant M. From these two remarks we obtain the following estimations on $| I_{kn} |$ and on $| L_n |$:

$$\begin{aligned}
| I_{kn} | &\leqslant \| H_\Phi - H_{Q_n\Phi Q_n} \| \, \|F_k(\cdot)\|_{\mathbf{H}^1_{C^1(\mathscr{E})}} \\
&\leqslant \| H_\Phi - Q_nH_\Phi Q_n \| \, M,
\end{aligned}$$

while

$$\begin{aligned}
| L_n | &\leqslant \| H_\Phi - H_{Q_n\Phi Q_n} \| \, \|F(\cdot)\|_{\mathbf{H}^1_{C^1(\mathscr{E})}} \\
&\leqslant \| H_\Phi - Q_nH_\Phi Q_n \| \, M.
\end{aligned}$$

Since the projections Q_n regarded as operators on $\mathbf{H}_{\mathscr{E}}{}^2$ converge strongly to the identity, and since H_Φ is a compact operator, the last expressions in the above inequalities converge to zero by Proposition 4.2. Thus, assertions (8.A) and (8.C) are proved.

The proof of (8.B) is analogous to the proof of the assertion (6.B) in the proof of Theorem III. It suffices to show that when regarded as a function in $\mathbf{L}^{\infty}_{\mathscr{L}(\mathscr{E}_n)}$, the function $Q_n\Phi(\cdot)Q_n$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E}_n)} + \mathbf{C}_{C^{\infty}(\mathscr{E}_n)}$. To prove that $Q_n\Phi(\cdot)Q_n$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E}_n)} + \mathbf{C}_{C^{\infty}(\mathscr{E}_n)}$, we let λ_n be the linear functional on $\mathbf{H}^1_{C^1(\mathscr{E}_n)}$ defined by the equation

$$\lambda_n\Big(G(\cdot) = \int_{\mathbf{T}} \operatorname{Tr}(Q_n\Phi(z)Q_nG(z)\Big)\,dm, \qquad G(\cdot)\in \mathbf{H}^1_{C^1(\mathscr{E}_n)},$$

and show that the kernel of λ_n is weak* sequentially closed.

Let $\{G_k(\cdot)\}_{k=1}^{\infty}$ be a sequence in the kernel of λ_n converging weak* to a function $G(\cdot)$ in $\mathbf{H}^1_{C^1(\mathscr{E}_n)}$. Using Theorem IV′ we factor each $G_k(\cdot)$ as $G_k(\cdot) = G_{1k}(\cdot)\,G_{2k}(\cdot)$, where each $G_{ik}(\cdot)$ is in $\mathbf{H}^2_{C^2(\mathscr{E}_n)}$ and

$$\|\,G_{ik}(\cdot)\|^2_{\mathbf{H}^2_{C^2(\mathscr{E}_n)}} = \|\,G_k(\cdot)\|_{\mathbf{H}^1_{C^1(\mathscr{E}_n)}}, \qquad i = 1, 2.$$

Since the sequence $\{G_k(\cdot)\}_{k=1}^{\infty}$ converges in the weak* topology on $\mathbf{H}^1_{C^1(\mathscr{E}_n)}$, it is bounded. Therefore the sequences $\{G_{ik}(\cdot)\}_{k=1}^{\infty}$ are bounded in $\mathbf{H}^2_{C^2(\mathscr{E}_n)}$, $i = 1, 2$, and, by passing to a subsequence if necessary, we may suppose that the sequences $\{G_{ik}(\cdot)\}_{k=1}^{\infty}$ converge to functions $G_i(\cdot)$, $i = 1, 2$, in the weak topology on $\mathbf{H}^2_{C^2(\mathscr{E}_n)}$. Imitating the proof of (6.B), it may be shown that $G(\cdot) = G_1(\cdot)\,G_2(\cdot)$. Using Lemma 8.2, we may write

$$\begin{aligned}\lambda_n(G_k(\cdot)) - \lambda_n(G(\cdot)) &= \sum_{j=1}^{n} (Q_nH_\Phi Q_nG_{1k}e_j\,,\,\tilde{G}_{2k}e_j)\\ &\quad - \sum_{j=1}^{n} (Q_nH_\Phi Q_nG_1e_j\,,\,\tilde{G}_2e_j).\end{aligned} \tag{8.2}$$

Using the facts that H_Φ is compact and that the sequences $\{G_{ik}(\cdot)\}_{k=1}^{\infty}$ converge weakly to $G_i(\cdot)$, $i = 1, 2$, it is easy to see that the sequence in Eq. (8.2) converges to zero, showing that $G(\cdot)$ is in the kernel of λ_n. This completes the proof of (8.B) and, with it, Theorem VII.

Notice that when $\mathscr{E}$ is one-dimensional, Theorem VII is Hartman's theorem [7].

The following theorem relates the commutant of a contraction to a class of generalized Hankel operators.

THEOREM VIII. *Let $\mathscr{E}$ be a separable (possibly finite-dimensional) Hilbert space, let $\Theta(\cdot)$ be an inner function in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$, let $\mathscr{H} = \mathbf{H}^2_{\mathscr{E}} \ominus \Theta_+\mathbf{H}^2_{\mathscr{E}}$, and let P be the projection of $\mathbf{H}^2_{\mathscr{E}}$ onto $\mathscr{H}$. Suppose $\Phi(\cdot)$ is in $\mathbf{H}^{\infty}_{\mathscr{L}(\mathscr{E})}$ and satisfies the relation $\Phi_+\Theta_+\mathbf{H}^2_{\mathscr{E}} \subseteq \Theta_+\mathbf{H}^2_{\mathscr{E}}$. If*

$S = P\Phi_+ \mid \mathscr{H}$ and if $f(\cdot)$ is in $\mathbf{H}^2_{\mathscr{E}}$, then $\| SPf \| = \| P\Phi Pf \| = \| H_{(U\Theta^*\Phi)} f \|$.

Proof. Let $g(\cdot)$ be in $\mathbf{H}^2_{\mathscr{E}}$ and let P_1 denote the projection of $\mathbf{L}^2_{\mathscr{E}}$ onto $\mathbf{H}^2_{\mathscr{E}}$; then

$$\begin{aligned}(H_{(U\Theta^*\Phi)} f, g) &= ((P_1 RU\Theta^*\Phi) f, g)\\ &= ((RU\Theta^*\Phi) f, g)\\ &= (\Theta^*\Phi f,\ U^*Rg).\end{aligned} \tag{8.3}$$

Since $g(\cdot)$ is in $\mathbf{H}^2_{\mathscr{E}}$, Rg is in $\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{0\mathscr{E}}$, and U^*Rg is in $\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}}$. Lemma 6.1 says, then, that $(\Theta^*\Phi f, U^*Rg) = (P\Phi Pf, \Theta(U^*Rg))$. Thus

$$(H_{(U\Theta^*\Phi)} f, g) = (P\Phi Pf, \Theta(U^*Rg)). \tag{8.4}$$

As $g(\cdot)$ ranges over the closed unit ball of $\mathbf{H}^2_{\mathscr{E}}$, U^*Rg ranges over the closed unit ball in $\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}}$, since U^* and R are unitary operators on $\mathbf{L}^2_{\mathscr{E}}$. Also, in the proof of Lemma 6.1, we saw that since Θ is a unitary operator on $\mathbf{L}^2_{\mathscr{E}}$, $\Theta(\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}}) = \mathscr{H} + (\mathbf{L}^2_{\mathscr{E}} \ominus \mathbf{H}^2_{\mathscr{E}})$. Therefore, as $g(\cdot)$ ranges over the unit ball of $\mathbf{H}^2_{\mathscr{E}}$, $(P\Theta U^*R)\, g$ ranges over the unit ball of $\mathscr{H}$. From this observation and Eq. (8.4) we have

$$\begin{aligned}\| H_{(U\Theta^*\Phi)} f \| &= \sup_{\substack{\|g\| \leqslant 1\\ g \in \mathbf{H}^2_{\mathscr{E}}}} |(H_{(U\Theta^*\Phi)} f, g)|\\ &= \sup_{\substack{\|g\| \leqslant 1\\ g \in \mathbf{H}^2_{\mathscr{E}}}} |(P\Phi Pf, \Theta(U^*Rg))|\\ &= \| P\Phi Pf \| = \| SPf \|.\end{aligned}$$

This last equation is the conclusion of Theorem VIII and the proof is complete.

To obtain Theorem III from Theorem VII, observe that Theorem VIII implies that S is compact if and only if $H_{(U\Theta^*\Phi)}$ is compact. By Theorem VII, $H_{(U\Theta^*\Phi)}$ is compact if and only if $U(\cdot)\, \Theta^*(\cdot)\, \Phi(\cdot)$ is in $\mathbf{H}^\infty_{0\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$; but this happens if and only if $\Theta^*(\cdot)\, \Phi(\cdot)$ is in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$. Thus S is compact if and only if $\Theta^*(\cdot)\, \Phi(\cdot)$ is in $\mathbf{H}^\infty_{\mathscr{L}(\mathscr{E})} + \mathbf{C}_{C^\infty(\mathscr{E})}$, and this is Theorem III.

Added in proof. Professor Sz.-Nagy has remarked to us that Lemma 3.3 is a consequence of the absolute continuity of the spectral measure for the minimal strong unitary dilation of T and the Riemann-Lebesgue Lemma.

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